## **Chapter 9 Responses to Periodic Inputs**

**9.1 Fourier Series**

* The defining property of a periodic function  of period *T* is that *f*(*t*) =  where  is any integer. That is, the function repeats every period *T*.
* A remarkable theorem on periodic functions is Fourier’s theorem:

***Statement*** *A periodic function can be expressed, in general, as an infinite series of cosine and sine functions:*

** (9.1.1)

*where  is a positive integer, and   and  are constants, known as* ***Fourier coefficients****, that depend on f*(*t*)*.*

* The component having  is the **fundamental**, whereas the component having *k* equal a particular integer *n* is the *n*th **harmonic**. The series expression of Equation 9.1.1 is the **Fourier series expansion** (FSE) of *f*(*t*).
* Combining the sine and cosine terms, the FSE becomes:

 (9.1.2)

where:   and  (9.1.3)

**9.2 Fourier Analysis**

***Summary*** *Given the four functions cosmω0t, sinmω0t, cosnω0t, and sinnω0t, where m and n are integers, the integral of the product of any two of these functions over a period T = 2π /ω0 is zero, except the products cos2nω0t, and sin2nω0t, having m = n, in which case the integral is T/2*:

 for all *n* and *m* (9.2.1)

 for  (9.2.2)

 (9.2.3)

* To determine  in the FSE, we integrate both sides of Equation 9.1.1 over a period:





or,  (9.2.4)

*  is the average of  over a period. It is the *dc component* of *f*(*t*), whereas the cosine and sine terms are the *ac component*.
  + To determine  we multiply both sides of Equation 9.1.1 by  and integrate over a whole period, invoking Equations 9.2.1 to 9.2.3:





This gives:

 (9.2.5)

* To determine  we multiply both sides of Equation 9.1.1 by  and integrate over a whole period, invoking Equations 9.2.1 to 9.2.3:





This gives:

 (9.2.6)

***Summary*** *a*0 is the average of *f*(*t*) over a period, *an* is twice the average of *f*(*t*)cosn*ω*0*t* over a period, and *bn* is twice the average of *f*(*t*)sin*ω*0*t* over a period.

**Example 9.2.1 FSE of Sawtooth Waveform**

It is required to derive the Fourier coefficients of the sawtooth waveform of Figure 9.2.1.

***Solution*:** During the interval  

;



. . The FSE does not have any cosine terms for reasons that will be explained below.

 , where *ω0T* = 2*π*. The trigonometric form of *fst*(*t*) is therefore:

 (9.2.7)

At the points of discontinuity, *t* = *kT*, where *k* is an integer. All the sinusoidal terms vanish and *f*(*t*) = *A*/2, the average of the values of  and .

**Exponential Form**

* The FSE can also be expressed in exponential form. It is convenient for this purpose to change the index  to *n*:



 (9.2.8)

* Let  Substituting for  and  from Equations 9.2.5 and 9.2.6:

 (9.2.9)

* It follows that:  and:

 (9.2.10)

where  is the complex conjugate of  Equation 9.2.8 can be expressed as:



* The last term on the RHS can be written in terms of negative values of  as:

 (9.2.11)

Equation 9.2.11 can be expressed more compactly as:

 (9.2.12)

* The relationships between   and  readily follow from the definition of *Cn*:

 and  (9.2.13)

 and  (9.2.14)



where *cn* and  are as in Equation 9.1.3.

**Frequency Spectrum**

* The plots of  and against frequency are, respectively, the **amplitude spectrum** and the **phase spectrum**

of *f*(*t*). They both constitute the

**frequency spectrum** of *f*(*t*).

* Because frequencies in the FSE have discrete values only, the frequency spectrum of a periodic function is a *line spectrum* that consists of a series of lines at , where  (Figure 9.2.2a).



* Since , it is seen that  and . The amplitude spectrum is an even function, whereas the phase spectrum is an odd function (Figure 9.2.2b), except when *Cn* is real, so *bn* = 0 and  is either zero or 180°.

**Example 9.2.2 Exponential Form of Sawtooth Waveform**

It is required to derive the exponential Fourier coefficients of the sawtooth waveform of Figure 9.2.1 and plot its amplitude and phase spectra.

***Solution*:** Integrating by parts (Appendix), noting that :

 (9.2.15)

 is imaginary, which means that  (Equation 9.2.13). The average value of *fst*(*t*) is *A*/2, and cannot be obtained by setting  in Equation 9.2.15.The exponential form of *fst*(*t*) is:

 (9.2.16)

The amplitude spectrum consists of a line of height *A*/2 at  and lines of height  at



; the phase angle of  is +90° for , and –90° for  (Figure 9.2.3).



*Cn* can be

obtained usingMatlab’s int(E,t,a,b) command. Ignoring for the moment , the integral  can be evaluated by entering the following code:

syms t n w

int(t\*exp(-j\*n\*w\*t),t,0,2\*pi/w)

Matlab returns a rather complicated expression. So enter: simplify(ans). Matlab returns:

1/n^2/w^2\*(2\*i\*exp(-2\*i\*pi\*n)\*n\*pi+exp(-2\*i\*pi\*n)-1)

Recognizing that 1 for all *n*, this expression simplifies to . Multiplying by  gives Equation 9.2.15.

If the function  of Figure 9.2.1 is negated (Figure 9.2.4a), then shifted upward by *A*, it becomes the ‘reversed sawtooth’ waveform  of Figure 9.2.4b. The FSE of  is obtained by adding *A* to the negation of the RHS of Equation 9.2.7:



 (9.2.17)

 of  is *A*/2 and its  is . The amplitude spectrum is unchanged, but the phase spectrum is negated. The derivation of the FSE of  from that of  illustrates a useful technique of deriving the FSE of a function from that of another function.

**Example 9.2.3 FSE of Rectangular Pulse Train**

It is required to derive the Fourier coefficients of the rectangular pulse train *fpt*(*t*) illustrated in Figure 9.2.5 and plot its amplitude and phase spectra.

***Solution*:** It is convenient to take a period that is symmetrical about the origin as shown. Hence,  in Equation 9.2.9, so that:











 (9.2.18)

where  When  is a nonzero integer, . However, according to L’Hopital’s rule, 

Since  is real, , and the FSE of *fpt*(*t*) function does not have any sine terms. The reason for this is that the function is even, as explained in the next section. The average value of  is  However, it can be obtained in this case by setting  in Equation 9.2.18.

If we set  and replace  by 2*π*:

 (9.2.19)

and  (9.2.20)

To express the FSE in trigonometric form, we substitute ,  and  in Equation 9.1.1:

 (9.2.21)

The amplitude spectrum is shown in Figure 9.2.6(a) for the case of  so that  for all integer values of  The amplitude is zero for  an integral multiple of  The lines are bounded by the dotted envelope representing the

function  for continuous . The



phase spectrum is shown in Figure 9.2.6b.

Since  is real, its phase angle is either

zero (), or (). The

phase angle is zero when *n* = 0,

since *C*0 is positive, and is

not defined if , as when *n*

= ± 5, because the

phase angle can be zero or 180° when

the magnitude is zero.

It can be readily verified that if , where  is a

positive (nonzero), odd integer, then  and . On the other hand, if  is a positive, even integer,  and  Thus, if  for  If  for , etc.

If *α* is small, it is seen from Equation 9.2.19 that all the harmonics have the same amplitude*αA*. This is an important result in signal analysis, according to which, the narrower the pulses, the more significant are the higher harmonics (Section 16.6).

To determine *Cn* using Matlab’s int(E, t, a, b) command, we enter the following:

syms t n w a



int(exp(-j\*n\*w\*t),t,-a/2,a/2)

simplify(ans)

Matlab returns: 2\*sin(1/2\*n\*w\*a)/n/w. Multiplying this by  gives Equation 9.2.18.

We can deduce from Equation 9.2.21

the FSE of a square wave of amplitude  and zero average value (Figure 9.2.7). To do so, we set   and remove the dc value by subtracting  from  Noting that  for even 

 (9.2.22)

**Translation in Time**

* If a periodic waveform  is delayed by  it becomes  with respect to the same time origin. Replacing  by  in Equation 9.2.12:

 (9.2.23)

* The effect is to replace  by  The magnitude of  and hence the amplitude spectrum, remains unchanged. However, the new phase angle  is:

 (9.2.24)

* Conversely, if the function is advanced by   is replaced by 

**Example 9.2.4 Translation in Time of Square Wave**

It is required to derive the FSE of the square wave (Figure 9.2.7) when delayed, or advanced, by *T*/4.



***Solution*:** Since *td* = *T*/4, *nω0td* = *nω0T*/4 = *nπ*/2. If the function is delayed by *T*/4 Figure 9.2.4, the phase angle of each of the terms in Equation 9.2.22 is decreased by *nπ*/2:

  (9.2.25)

If the square wave of Figure 9.2.7 is advanced by *T*/4, it becomes the negation of Figure 9.2.8, so that:

 (9.2.26)

**9.3 Symmetry Properties of Fourier Series**

**Even-Function Symmetry**

***Concept*** *The FSE of an even periodic function does not contain any* *sine terms; its Fourier coefficients can be evaluated over half a period.*

* The reason is that, since the sine function is odd, the presence of sine terms introduces odd components in the function and destroys its even symmetry.
  + Examples of even functions are the rectangular pulse train (Figure 9.2.5) and the square pulse of Figure 9.2.7. The corresponding FSEs (Equations 9.2.21 and 9.2.22) do not have any sine terms.
* If the FSE of an even periodic function does not contain any sine terms, then *bn* = 0 and *Cn* is real. Since a period of an even periodic function is centered about the vertical axis, *Cn* can be expressed as:

 (9.3.1)

If we substitute  in the first integral in brackets, this integral becomes . Changing the dummy integration variable back to *t* and invoking the property of an even function that *f*(*t*) = *f*(-*t*), the integral becomes . Substituting in Equation 9.3.1, combining with the second integral, and m

aking use of the relation , we obtain:

=  (9.3.2)

* It follows that for an even function:

 , and  for all *n* (9.3.3)

**Odd-Function Symmetry.**

***Concept*** *The FSE of an odd periodic function does not contain an average term nor any* co*sine terms; its Fourier coefficients can be evaluated over half a period.*

* The reason that the FSE of an odd periodic function does not contain an average term nor any cosine terms is that these terms, being even, introduce even

components and destroy the odd symmetry of the function.

* An example of an odd function is the square wave of Figure 9.3.1. The FSE (Equation 9.3.1) consists of sine terms only.
* A function that appears to be neither odd nor even can become odd when the dc component is removed. An example is the sawtooth waveforms of Figs. 9.2.1 and 9.2.4. If the dc component *A*/2 is subtracted, the function becomes odd. Hence, a function can have an odd ac component but is not odd because of a dc component.
* If the FSE of an odd periodic function does not contain any cosine terms, *an* = 0 and *Cn* is imaginary. Pursuing an argument analogous to that above for an even function, it follows that for an odd periodic function:

, and =  (9.3.4)

or,  for all *n*, and  (9.3.5)

**Half-Wave Symmetry**

* A periodic function has half-wave symmetry if:

, or  (9.3.6)

***Concept*** *The FSE of a half-wave symmetric periodic function does not contain an average term nor any* *even harmonics; its Fourier coefficients can be evaluated over half a period. Thus*:

, for *n* odd

*and* , for *n* even or zero (9.3.7)

* To prove this property, we express *Cn* as:  . Substituting  the second integral becomes: . Changing the dummy variable  back to , invoking the half-symmetry property, and substituting , the integral becomes: . But  for odd  and  for

even or zero. Equations 9.3.7 then follow.

* In terms of the coefficients a and b of the trigonometric form:

,  for *n* even, and:

,  for *n* odd (9.3.8)

* The square wave of Figure 9.2.7 is both even and half-wave symmetric; its FSE (Equation 9.2.22) consists of odd cosine terms only. The square wave of Figure 9.2.8 is both odd and half-wave symmetric; its FSE (Equation 9.2.25) consists of odd sine terms only. The waveform of Figure 9.3.1 is half-wave symmetric but is neither odd nor even.



**Quarter-Wave Symmetry**

* A *half-wave symmetric* function that is also symmetrical about a vertical line through the middle of the positive or negative half cycles is said to possess **quarter-wave symmetry**. Such a function can always be made either odd or even by translating it in time. The square waves of Figs. 9.2.7 and 9.2.8 are examples.
* The FSE of an odd, quarter-wave symmetric function consists of odd sine terms only, so that:

,  for all *n*,  for even *n*

* *bn* for odd *n* need only be evaluated over a quarter period, from *t* = 0 to = *t* = *T*/4:

 for odd *n* (9.3.9)

This is because both *f*(*t*) and sin*nω0t*, with *n* odd, are symmetrical about the middle of the half-cycle from *t* = 0 to *t* = *T*/2.

* Similarly, the FSE of an even, quarter-wave symmetric function consists of odd cosine terms only, so that:

,  for all *n*,  for even *n*

* *an* for odd *n* need be evaluated over a quarter period only:

 for odd *n* (9.3.10)

Again, this because both *f*(*t*) and cos*nω0t*, with *n* odd, are symmetrical about the

middle of the half-cycle from *t* = 0 to = *t* = *T*/2.

**Table 9.3.1 Summary of Symmetry Properties of Periodic functions**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Type of Symmetry** | | | ***bn*** | ***a*n** | ***a*0** |
| Neither odd nor even | | |  |  |  |
| Even | | | 0 |  |  |
| Odd | | |  | 0 | 0 |
| Half-wave Symmetry | Neither odd nor  Even | | *n* odd, 0 for *n* even | *n* odd, 0 for *n* even | 0 |
| Quarter-wave Symmetry | Even | 0 | *n* odd, 0 for *n* even |
| Odd | *n* odd, 0 for *n* even | 0 |

**Example 9.3.1 FSE of Triangular Waveform**

It is required to determine the FSE of the triangular waveform of Figure 9.3.2.

***Solution*:** The function has zero average, is even, and possesses half-wave symmetry. It is also quarter-wave symmetric. Its FSE must contain odd cosine terms only. Over the

interval   It follows from Equation 9.3.10 that:



=

 (9.3.11)

where using the exponential form makes the integration by parts somewhat simpler. It

follows that . In evaluating this expression, *even values of n should not be used*, because in applying Equation 9.3.11, we have already restricted *n* to be odd on account of half-wave symmetry. Hence,

 where  is odd (9.3.12)

The FSE of  is therefore:

 (9.3.13)

**9.4 Derivation of FSEs from those of Other Functions**

**Addition/Subtraction/Multiplication.**

***Concept*** *The FSEs of some functions can be derived from FSEs of other functions having the same period, through addition, subtraction, or multiplication.*

**Example 9.4.1 FSE of Half-Wave Rectified Waveform**



It is required to determine the FSE of: (a) the half-wave rectified waveform of Figure 9.4.1a; and (b) the full-wave rectified waveform of Figure 9.4.1b.

***Solution*:** (a) The given half-wave rectified waveform can be considered to be the product of a cosine function of amplitude 



and a square pulse train of unity amplitude, both functions having the same period  (Figure 9.4.2). The FSE of the pulse train is that of Equation 9.2.21, with *A* = 1 and

 The FSE of the cosine function is the function itself. Hence:



 



, *n* = 1, 2, 3, … (9.4.1)

The FSE contains a dc component of *A*/*π*, a fundamental component *A*/2, and even harmonics as cosine terms, as to be expected of an even function.

(b) The FSE of the full-wave rectified waveform of Figure 9.4.1b may be derived by considering it as the sum of a half-wave rectified waveform of amplitude  and the function . When is added to the RHS of Equation 9.4.1 multiplied by 2, the *ω*0*t* term cancels out, giving the FSE for full-wave rectified waveform:

*ffw*(*t*) ,

*n* = 1, 2, 3, … (9.4.2)

Note that the lowest allowed angular frequency is.The FSE could be expresses in terms of . The ac component of the FSE then consists of a

fundamental of frequency  and odd and even harmonic of this frequency.

The full-wave rectified waveform could also be considered as: i) the sum of two half-wave rectified waveforms, with one waveform shifted by half a period with respect to the other waveform; ii) the product of a square wave of zero average and 

**Differentiation/Integration.**

***Concept*** *The FSE of a given periodic function can be differentiated, or integrated, term by term. The result is the FSE of a periodic function that is the derivative, or integral, of the given function, except that integrating the dc component destroys the periodicity of the function.*

* This follows from differentiating, or integrating, both sides of the FSE.
* When a periodic function having a dc component is differentiated, the dc component vanishes and the resulting function is periodic with zero average. But when a periodic function having a dc component is integrated, the integral of the dc component increases linearly with time, which destroys the periodicity of the function, although the integral of the original ac component is still periodic.

**Example 9.4.2 Integral of FSE**

It is desired to obtain the FSE of the triangular waveform as the integral of the square waveform.

***Solution*:** Consider the delayed square waveform of Figure 9.2.8, whose FSE is given by Equation 9.2.25. Integrating this FSE gives:

 (9.4.3)

where the constant of integration is the average value of the function. The RHS of Equation 9.4.3, with  is identical with  (Equation 9.3.13), bearing in mind that the peak-to-peak amplitude of the triangular wave equals the area under one half-cycle of the square

wave. Thus,  Hence,  as in Equation 9.3.13.

**Rate of Attenuation of Harmonics**

* The more rapidly the magnitudes of the harmonics decrease with the order of the harmonic, the fewer are the number of terms of the FSE that have to be included to obtain a given degree of accuracy. The rate of attenuation of harmonics is related to the degree of ‘smoothness’ of the function:

***Concept*** *The smoother the function, the more rapidly the harmonics decrease in magnitude.*

* The smoothness of a function depends on the continuity of the function and its higher derivatives: If the *m*th derivative of a periodic function is discontinuous, with all the

derivatives of lower order being continuous, the magnitudes of the harmonics decrease approximately as   the derivative of order 0 being the function itself.

* + This is exactly true of the square and triangular waveforms. It is approximately true of the pulse train, the half-wave and full-wave rectified waveforms.

**9.6 Circuit Responses to Periodic Functions**

**Concept** The steady-state response of an LTI circuit to a periodic signal is the sum of the responses to each component acting alone.



* Let in Figure 9.6.1 be a general periodic function of the form of Equation 9.1.2:

 (9.6.1)

* Since the ac components of the input are sinusoids, the steady-state output due to each of these components can be determined by phasor analysis. For the *n*th harmonic, we have, from voltage division:

 (9.6.2)

* For *ω*0 = 0, the circuit is a simple voltage divider, and . The *n*th
* harmonic in the output has a magnitude that is  that of the corresponding input component and lags this component by  The FSE of the output is therefore:



 (9.6.3)

**Example 9.6.1 FSE of Response of *RC* Circuit to a Square Wave Input**

The square wave of Figure 9.2.8 is applied to the *RC* circuit of Figure 9.6.2. It is required to determine the output *vO*.



***Solution*:** The FSE of the input voltage is given by Equation 9.2.26, with  replaced by . According to Equation (9.5.2), with  and  the *n*th harmonic in the output has a magnitude that is  that of the corresponding input component and lags this component by  The FSE of the output can therefore be expressed as:

  (9.6.4)

* 1. **Average Power and rms Values**

***Concept*** *In an LTI circuit, components of different frequencies do not interact, and the total average power is the sum of the average powers due to each component acting alone.*

* To justify this, consider a periodic input voltage  of the form:

 (9.7.1)

to be applied to two terminals of an LTI circuit. The input current  at these terminals is also periodic, of the same frequency, and can be expressed as:

 (9.7.2)

* The average power input to the circuit is:

 (9.7.3)

* According to Equations 9.2.1 to 9.2.3, the product terms involving trigonometric functions evaluate to zero over *T*, except those having the same  Thus:



 (9.7.4)

* The integral of the second term in the integrand vanishes over a period *T*, so that:

 (9.7.5)

* It is seen from Equation 9.7.5 that the average power is due only to components of voltage and current of the same frequency. The average power of each frequency component is given by the product of the rms voltage, the rms current, and the power factor cos(*θvn – θin*), as in Equation 7.1.10.
* Components of different frequencies do not contribute to the average power. They do contribute to the instantaneous power, as does the component of frequency 2*nω*0 in Equation 9.7.4, causing a net power flow in or out of the circuit at any instant. However, this power averages to zero over a complete cycle, leaving only the contribution from components having the same frequency.

**rms Value**

* By definition,  the rms value of  is given by:

 (9.7.6)

* Assuming  to be given by Equation 9.1.2, and again using Equations 9.2.1 to 9.2.3, it follows that:







or  (9.7.7)

* According to Equation 9.7.7, the rms value of a periodic function is the square root of

the sum of the squares of the rms values of the individual components.

* In terms of  and *b* coefficients (Equation 9.1.3), Equation 9.7.7 becomes:

 (9.7.8)

* If a periodic voltage  is applied across a resistor  the rms current component  corresponding to a voltage component  is  With , It follows from Equation 9.7.5 that:

 (9.7.9)

and,  (9.7.10)

* If a periodic waveform  is expressed analytically, it is usually much simpler to determine its rms value from direct application of Equation 9.7.6 rather than from its Fourier coefficients (Eq. 9.7.7).
  + In the case of the half-wave rectified waveform of Figure 9.4.1a, the mean of its square over a period is:  Hence, the rms value is  The dc component is  (Eq. (9.4.1)). It follows from Eq. (9.7.7) that:  This gives for the ac component of the half-wave rectified waveform:  0.39*A*.

**Example 9.7.1 rms Value of Periodic Triangular Waveform**

It is required to determine the rms value of the periodic triangular waveform  shown in Figure 9.7.1 and deduce the rms value of the ac component.



***Solution*:** For   The

square of this waveform is  and the area under the curve is:

 (9.7.11)



For  the area under the curve of the squared function is clearly the same as that for  in Figure 9.7.2. By analogy with Eq. (9.7.23), the area under the square function is , . The total squared area for one period of  is the sum  The mean is  and the rms value is  Because it is independent of *τ*, this result applies to any triangular waveform that varies between  and  and repeats continuously without interruption, including a sawtooth waveform 

The dc, or average value of  is  If the rms value of the ac component is denoted by  then  This gives  